Section B, Question 4

a) Amplification/Attenuation: ensure that the amplitude of the input signal fits to the range of the ADC, i.e. clipping or or insufficient use of the quantiser characteristic by very small input amplitudes is avoided.
Anti-alias filtering: remove frequencies above $f_s/2$ in order to avoid aliasing in the sampling stage.

b) Assume a quantiser with range $[-V;V]$. Therefore with B bits, $q = 2V2^{-B} = V2^{1-B}$. The signal power of a sinusoid with amplitude V is $V^2/2$. Thus,

$$\text{SQNR} = \frac{V^2/2}{q^2/12} = \frac{V^2 12}{V^2 2^{2-2B}} = 3 \cdot 2^{2B} \tag{1}$$

and

$$\text{SQNR}_{\log} = 10\log_{10} 3 \cdot 2^{2B} = 10\log_{10} 3 + 20B\log_1 0(2) \approx 2 + 6B[\text{dB}] \tag{2}$$

(note: $10\log_{10} 2 = \log_{10} 1024 \approx 3$)

c) Assume oversampling by a factor of N. There are 4 stages: (i) anti-alias filter with slow roll-off to a cut-off at $N * f_2/2$; (ii) sampling at $N * f_s$; while the signal of interest is situated in the baseband region, the quantisation is spread flat over the entire spectrum; (iii) by lowpass filtering, the noise can be suppressed above $f_s/2$ using a digital filter; this reduces the noise power by a factor of N; (iv) the sampling rate is adjusted by decimation by a factor of N. The reduced SQNR appears as if a virtually higher bit resolution had been used in the quantisation: every oversampling factor of 4 yields one extra bit.

d) Analysis:

$$\begin{aligned} Y(z) &= N(z) + H(z)\left\{H(z)\left(X(z) + z^{-1}Y(z)\right) + z^{-1}Y(z)\right\} && (3)\\ Y(z) & \left\{1 + H(z)H(z)z^{-1} + H(z)z^{-1}\right\} = N(z) + H(z)H(z)X(z) && (4) \end{aligned}$$

with

$$1 + \frac{z^{-1}}{(1-z^{-1})^2} + \frac{z^{-1}}{1-z^{-1}} = \frac{(1-z^{-1})^2 + z^{-1} + (1-z^{-1})z^{-1}}{(1-z^{-1})^2} = \frac{1}{(1-z^{-1})^2} \tag{5}$$

and thus

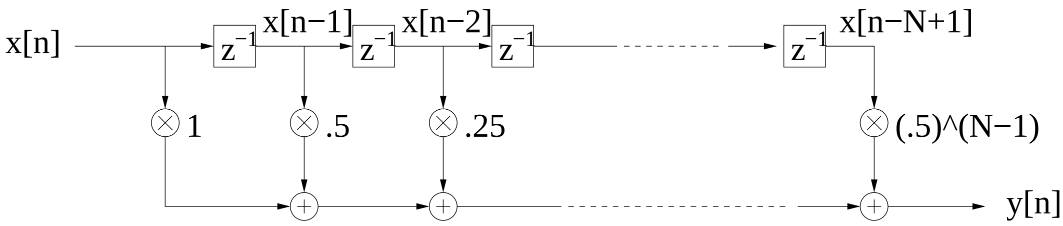
$$Y(z) = \frac{1}{(1-z^{-1})^2}N(z) + X(z) \tag{6}$$

Therefore, $H_\text{X}(z) = 1$ and $H_\text{N}(z) = (1-z^{-1})^2$.

e) $|H_\text{X}(e^{j\Omega})| = 1$;

$$\begin{aligned} |H_\text{N}(e^{j\Omega})| &= \sqrt{H_\text{N}(e^{j\Omega})H_\text{N}^*(e^{j\Omega})} && (7)\\ &= \sqrt{(1-e^{-j\Omega})^2(1-e^{j\Omega})^2} = (1-e^{-j\Omega})(1-e^{j\Omega}) && (8)\\ &= 2 - 2\cos\Omega && (9) \end{aligned}$$

f) Σ-Δ exploits oversampling and noise shaping: while the transfer function for the signal of interest (located in the baseband) is flat, the quantisation noise is highpass filtered and therefore disemphasised in the baseband.



a)

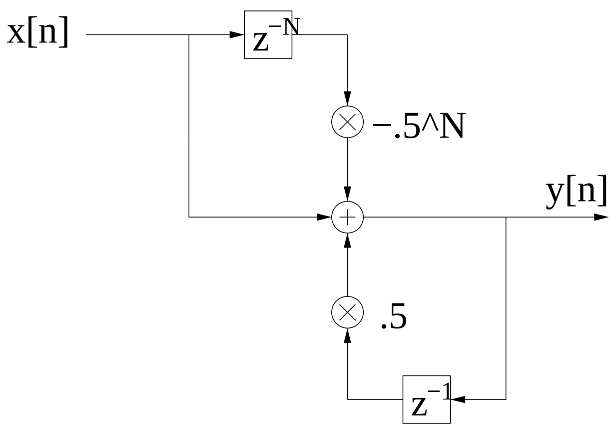
b)
   i) FIR filter with impulse response equal zero for $n \geq N$.
   ii) Causal since impulse response zero for $n < 0$.
   iii) Non-linear phase due to non-symmetric impulse response.
   iv) Minimum phase due to monotonously decreasing coefficients.

c) z transform:

$$H(z) = Y(z)/X(z) = \sum_{k=0}^{N-1} (\frac{1}{2})^k z^{-k} \quad (10)$$

d) Using geometric series:

$$\begin{aligned} H(z) &= \sum_{k=0}^{\infty} (\frac{1}{2})^k z^{-k} - \sum_{k=N}^{\infty} (\frac{1}{2})^k z^{-k} & (11) \\ &= \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{2})^N z^{-N} \sum_{k=0}^{\infty} (\frac{1}{2})^k z^{-k} & (12) \\ &= \frac{1 - \frac{1}{2})^N z^{-N}}{1 - \frac{1}{2} z^{-1}} & (13) \end{aligned}$$



e) FIR implementation requires $N-1$ multiply accumulates (note unity of first coefficient), IIR implementation requires 2 multiply accumulates. Hence for $N = 3$ both system have same complexity and for $N > 3$ the IIR realisation requires less multiply accumulates.

f)
   i) IIR in general requires considerably less coefficients. Note that from geometric series considerations above, an FIR systems takes considerably more MACs than IIR due to infinite number of zeros required to mimick a pole.
   ii) IIR filters, particularly of higher order, tend to a large dynamic range for their coefficients; hence quantisation in a fixed point implementation can affect small coefficients; in some IIR filters may even become unstable when quantised. The stability o FIRfilters is not affected by quantisation.

iii) FIR filters are guaranteed to be linear phase if they have a symmetric impulse response; IIR filters are never be linear phase. However, IIR filters can be designed to have at least an approximately linear phase response in their passband region.

Section B, Question 6: Optimal and Adaptive Filtering

a) An adaptive filter will produce an output from a provided input signal such that, when subtracted from a desired signal, the resulting error will be minimised in a suitable sense.

b) For $W(z) \bullet\!\!-\!\!\circ\, w[n]$ by either polynomial division or geometric series:

$$W(z) = \frac{1}{C(z)} = \sum_{k=0}^{\infty}(-\frac{1}{2})^k z^{-k} \tag{14}$$

c) Wiener-Hopf:

$$\mathbf{w}_{\text{opt}} = \mathbf{R}_{xx}^{-1}\mathbf{p} = \mathbf{U}\begin{bmatrix} .25 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & 100 \end{bmatrix}\mathbf{U}^{\mathrm{H}}\mathbf{p} \tag{15}$$

The covariance matrix is ill-conditioned due to a near-zero eigenvalue; hence in the inversion of $\mathbf{R}_{xx}$, any inaccuracies in the estimation will be considerably amplified and may disturb the performance considerably.

d) Noise covariance matrix: $\mathbf{R}_{nn} = \mathbf{I}$. Therefore noisy covariance matrix:

$$\mathbf{R}_{xx} = \mathbf{U}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}\mathbf{U}^{\mathrm{H}} + \mathbf{I} = \mathbf{U}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1.01 \end{bmatrix}\mathbf{U}^{\mathrm{H}} \tag{16}$$

due to $\mathbf{U}\mathbf{U}^{\mathrm{H}} = \mathbf{I}$. This matrix is no longer ill-conditioned due to regularisation by the noise. The Wiener-Hopf solution now does not represent a perfect inversion anymore, but a trade-off between inverting $C(z)$ over its range of excitation by the input signal, and between the amplification of the channel noise $v[n]$.

e) With a tap delay line vektor $\mathbf{x}[n]$ and a coefficient vector at time n, $\mathbf{w}[n]$, the squared error is given by

$$\xi = e^2[n] = d[n] - \mathbf{w}^{\mathrm{T}}\mathbf{x}[n] \quad . \tag{17}$$

LMS is a stochastic gradient technique:

$$\mathbf{w}[n+1] = \mathbf{w}[n] - \mu\nabla\xi \tag{18}$$

$$\nabla\xi = \frac{\partial}{\partial\mathbf{w}^{\mathrm{T}}}e^2[n] = 2e[n](-\mathbf{x}[n]) \tag{19}$$

f) This is a maximum phase system whose inverse is unstable. With delay Δ:

$$W(z) = \frac{z^{-\Delta}}{C(z)} = \frac{z^{-\Delta+1}}{1+\frac{1}{2}z} = z^{-\Delta+1}\sum_{k=0}^{\infty}(-\frac{1}{2})^k z^k \tag{20}$$

$$= \underbrace{z^{-N+1} - \frac{1}{2}z^{-N+2} + \cdots + (\frac{1}{2})^{N-1}}_{\text{causal}} + \underbrace{(\frac{1}{2})^N z + (\frac{1}{2})^{N+1}z^2 + \cdots}_{\text{anti-causal}} \tag{21}$$

The adaptive filter can identify, and converge to, the causal part.