Coursework Assignment for Signal Processing

- Digital Filtering:

  Given is the transfer function $H(z) = 1 + \frac{5}{2}z^{-1} + z^{-2}$.

  1. **Q:** Without explicitly calculating the magnitude response, state the filter gain for $\Omega = 0$ and $\Omega = \pi$.

     A:

     $$\begin{aligned} H(e^{j0}) &= \sum_{n=0}^{2} h[n] = -\frac{9}{2} \\ H(e^{j\pi}) &= \sum_{n=0}^{2} h[n](-1)^n = -\frac{1}{2} \end{aligned}$$

  2. **Q:** What are the phase properties: is $H(z)$ linear or non-linear phase? Further, is it a minimum, non-minimum, or maximum phase system?

     A: Due to its symmetric impulse response, $H(z)$ must be linear phase. Therefore, it can be neither minimum nor maximum phase but has to be non-minimum phase.

  3. **Q:** Calculate the frequency response $H(e^{j\Omega})$ to $H(z)$.

     A:

     $$\begin{aligned} H(e^{j\Omega}) &= 1 + \frac{5}{2}e^{-j\Omega} + e^{-j2\Omega} \\ &= e^{-j\Omega}\left(\frac{5}{2} + 2\cos(\Omega)\right) \\ &= (\frac{1}{2} + 4\cos^2(\frac{\Omega}{2}))e^{-j\Omega} \end{aligned}$$

     Hence magnitude and phase responses are $|H(e^{j\Omega})| = \frac{1}{2} + 4\cos^2(\frac{\Omega}{2})$ and $\angle\{H(e^{j\Omega})\} = -\Omega$, respectively.

  4. **Q:** State a minimum phase system $H_{\text{min}}(z)$ and a maximum phase system $H_{\text{max}}(z)$ that have the same magnitude response as $H(z)$.

     A: Note that $H(z) = (1 + \frac{1}{2}z^{-1})(1 + 2z^{-1})$. Therefore,

     $$H_{\text{min}}(z) = (1 + \frac{1}{2}z^{-1})(2 - z^{-1}) = 2(1 + \frac{1}{2}z^{-1})^2$$

     and

     $$H_{\text{max}}(z) = (\frac{1}{2} - z^{-1})(1 + 2z^{-1}) = \frac{1}{2}(1 + 2z^{-1})^2$$

     both have the same magnitude responses as $H(z)$.

  5. **Q:** Given the channel transfer function $C(z) = 1 + 2z^{-1}$. Calculate a (not necessarily causal but stable) equaliser $W(z) = C^{-1}(z)$.

     A: The direct inverse $W(z) = \frac{1}{1+2z^{-1}}$ is unstable. Exploiting the geometric series expansion,

     $$\begin{aligned} C(z) &= \frac{1}{1 + 2z^{-1}} = \frac{z}{2}\frac{1}{1 + \frac{1}{2}z} \\ &= \frac{z}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^n \\ &= \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n z^n \qquad (1) \end{aligned}$$

     is anti-causal but stable.

6. **Q:** Relaxing the equaliser to $W(z) = z^{-10}C^{-1}(z)$, find an FIR implementation of the causal part of $W(z)$. (**Note the error in the assignment sheet: $W(z)$ and $C(z)$ were swapped**)

   A: New equaliser:

$$
\begin{aligned}
W(z) &= z^{-10}\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n z^n \\
&= \sum_{1}^{\infty}\left(\frac{1}{2}\right)^n z^{n-10} \\
&= \underbrace{\sum_{n=1}^{\infty}(\frac{1}{2})^{-10-n}z^n}_{\text{anti-causal}} + \underbrace{\sum_{n=0}^{9}(\frac{1}{2})^{10-n}z^{-n}}_{\text{causal}}
\end{aligned}
$$

   Therefore,

$$
W_{\text{causal}}(z) = \left(\frac{1}{2}\right)^{10} + \left(\frac{1}{2}\right)^{9} z^{-1} + \cdots + \left(\frac{1}{2}\right)^{2} z^{-8} + \left(\frac{1}{2}\right) z^{-9}
$$

- Adaptive Digital Filtering:

1. **Q:** Using a labelled generic adaptive filter diagram, derive the LMS algorithm.

   A: Your answer should include a filter block with an input $x[n]$ and output signal $y[n]$; the (time-varying) coefficients are held in a vector $\mathbf{w}[n]$. The output is subtracted from a desired signal $d[n]$, forming the error signal $e[n]$.

   Derivation: based on the quadratic nature of the mean square error cost function, the LMS is a stochastic gradient algorithm, updating its coefficients at every time instance in the direction of the cost function's gradient:

$$
\mathbf{w}[n+1] = \mathbf{w}[n] - \mu\nabla\hat{\xi}_{\text{MSE}} \tag{2}
$$

   whereby the gradient is calculated as

$$
\begin{aligned}
\nabla\hat{\xi}_{\text{MSE}} &= \frac{\partial}{\partial\mathbf{w}[n]}e^2[n] = 2e[n]\,\frac{\partial}{\partial\mathbf{w}[n]}e[n] \\
&= 2e[n]\,\frac{\partial}{\partial\mathbf{w}[n]}(d[n] - \mathbf{w}^{\mathrm{T}}[n]\mathbf{x}[n]) \qquad (3)\\
&= -2e[n]\,\mathbf{x}[n] \qquad (4)
\end{aligned}
$$

   where

$$
\mathbf{x}[n] = \begin{bmatrix} x[n] \\ x[n-1] \\ \vdots \\ x[n-N+1] \end{bmatrix}
$$

   and step (3) assumes that the coefficient vector is independent of the desired signal.

   Inserting (4) into (2) gives the LMS update.

2. **Q:** Write down a short Matlab programme to simulate the LMS algorithm to identify an "unknown" system $C(z) = 1 - \frac{1}{2}z^{-2} + \frac{1}{3}z^{-3}$. Your adaptive filter should have four coefficients, and you should select an appropriate step size. Your input signal should be about 1000 samples long and consist of (i) Gaussian white noise and (ii) a sinusoid with normalised angular frequency $\Omega = \pi/2$.

   For cases (i) and (ii), provide plots of the instantaneous squared error. For case (ii), provide a justification for the adaptive filter coefficient after convergence (hint: study the frequency response).

   A: Example for the code using a white Gaussian input:

```
x = randn(1,1000);                          % input signal
c = [1 0 -1/2 1/3];                           % ‘‘unknown’’ system
d = filter(c,1,x);                          % desired signal
L = 10;                                     % filter length
mu = 0.01;                                  % step size

w = zeros(L,1);                             % weight vector initialisation
for n = L:1000,
    Xtdl = x(n:-1:n-L+1)’;                  % update TDL
    y(n) = w’*Xtdl;                         % calculate filter output
    e(n) = d(n)-y(n);                       % calculate error
    w = w + 2*mu*e(n)*Xtdl;                 % LMS weight update
end;
```

For case (ii), the input would be

```
x = sin(0.5*pi*(1:1000));
```

Convergence curves and magnitude plots:

As evident form the figure, the white noise excitation leads to a complete identification of the unknown system across the frequency range. In the case of sinusoidal input, the correct gain is only adjusted at $\Omega = \frac{\pi}{2}$, where the input signal has energy (“persistent excitation”). The same could be observed for the phase, which is only correct in the point $\Omega = \frac{\pi}{2}$.

3. **Q:** Why would practitioners prefer the LMS over the implementation of the Wiener-Hopf solution?

- Data Conversion

1. **Q:** The preconditioning stage of an ADC is adjusted such that the mean input power is 10% of the power of a sinusoid extending over the full input range of the quantiser characteristic, in order to avoid potential clipping in the quantisation stage. How must the word length R be selected to guarantee an SQNR of at least 63 dB?

   A: Let the range be $[-V; V]$. The power of a sinusoid is $V^2/2$ (e.g. $\frac{1}{2\pi}\int_0^{2pi} V^2 \cos^2 t dt = \frac{V^2}{2\pi}\int_0^{pi}(1+\cos 2t)dt = \frac{V^2}{2}$). Therefore the ADC input signal has a power of $\frac{V^2}{20}$.
   Noise power is $q^2/12$, whereby $q = 2V/2^R$.
   Therefore,

$$\text{SQNR} = \frac{V^2}{20}\,\frac{12\;2^{2R}}{4V^2} = \frac{3\;2^{2R}}{20}$$

and

$$\text{SQNR}_{dB} = 10\left(\log_{10}\frac{3}{20} + 2R\log_{10}2\right) = -8.24 + 6R$$

Thus, $R = 6R - 8.24 \geq 63$ implies $6R \geq 71.24$ or $R \geq 12$.

2. **Q:** An audio signal should be sampled at 16kHz. The ADC operates at 256kHz and is followed by a digital postprocessing stage comprising a digital anti-alias filter and a decimator. Sketch the requirement for the analogue anti-alias filter at the ADC input.

   A: The AAF has a passband from 0-8kHz, the transition band extends from 8kHz to (256-8)kHz = 248kHz, which permits the use of a gentle roll-off, and therefore an "inexpensive" AAF design and implementation compared to direct sampling at 16kHz.

3. **Q:** Assuming an ideal digital anti-alias filter, calculate and compare the SQNR for the system in (2) with an ADC sampling directly at 16kHz. How would the wordlengths be selected to achieve the same SQNR?

   A: Assume that in both cases we quantise with R bits, resulting in a noise power of $q^2/12$ spread over the spectrum $[0; 2\pi)$. In the oversampled case, most of the noise is outside the band of interest and can be suppressed by a digital lowpass filter cutting off at 8kHz. Thus, only $\frac{1}{16}$th of the noise remains.
   The noise power in the oversampled case:

$$\frac{1}{16}\frac{q^2}{12} = \frac{(q/4)^2}{12} \tag{5}$$

   suggests an effectively 4 times finer quantier characteristic equivalent to a quantisation with $R+2$ instead of R bits resolution as achieved in the standard (16kHz sampled) case.

4. **Q:** Derive the equivalent digital model of a first order $\Sigma - \Delta$ device, such that $Y(z) = X(z) + (1 - z^{-1})N(z)$ for an ADC with input $X(z)$, uncorrelated observation noise $N(z)$ and output $Y(z)$. Calculate the PSD of the quantisation noise at the output of the device, and explain how technique can extract resolution in addition to oversampling.

   A: